"New Approach to Study of Qualitative Analysis of Fractional Difference Equations and its Applications"

Synopsis Submitted to

Swami Ramanand Teerth Marathwada University, Nanded, Maharashtra, India

Research proposal submitted in partial fulfilment of registration to Ph.D. Program

DOCTOR OF PHILOSOPHY

In the Subject of

MATHEMATICS

Under the Faculty of

Science and Technology

Submitted by

Mr. Yogesh H. Shirole

yogeshshirole2611@gmail.com

Under the Supervision of

Dr. S. M. Jogdand

Department of Mathematics
S. S. G. M. Mahavidyalaya, Loha
Dist - Nanded - 431509, (MH), India
smjog12@gmail.com

Year: 2022

"New Approach to Study of Qualitative Analysis of Fractional Difference Equations and its Applications"

I. Introduction to proposed research topic:

This synopsis gives a brief description of the literature carried out towards execution of the research work embedded in this thesis in the present form. In the subsequent part of the section, we have mentioned the review process, the objective outlines and the problem statement of our study. Fractional differential equation is a generalization of ordinary differential equations and integration to arbitrary non-integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventieth century. It is widely and efficiently used to describe many phenomena arising in engineering, physics, economy, and science.

Difference equations are meant to describe some property or phenomenon that change over time. In contrast to differential equations, difference equations deal with discrete time or sequential discrete free variables in general. Many problems are discrete by nature like population size over successive generations, machine state over several iterations, or a physical phenomenon in successive time instances or it can be used in an approximation of a continuous phenomenon for easier analysis and the possibility of using computer simulations, etc. Difference equations can prove to be an extremely powerful (and sometimes the only) tool in handling many complex problems. They find it used frequently in stability, asymptotic, oscillation theories, and system control theory. In turn, these fields affect many other disciplines from biology and economy to physics and computer science. The Elaydi [1] introduces the concept of difference equations along with several tools, as well as many example applications. The book is intended as an introduction, and it does not require any prior knowledge of difference equations. However, some knowledge of differential equations will make it easier to relate to the discussions. Also, it is a prerequisite to be familiar with ordinary calculus and linear algebra. Some parts of the book will need some additional knowledge of complex analysis and advanced calculus. However, in most parts, these parts can be skipped if they are not of main interest to the reader. V. Lakshmikantham [2] introduce iterative processes and numerical methods for difference equations and theory of stability of Lyapunov including converse theorems and total and practical stability. Moreover, discussed the application of the theory of difference equations to numerical analysis.

The theory and applications of difference equations are found to more useful in the engineering field. Agarwal [3], Kelley and Peterson [4] developed the theory of difference equations and difference inequalities. Diaz and Osler published their 1974 paper [5] introducing a discrete fractional difference operator defined as an infinite series, a generalization of the binomial formula for the Nth-order difference operator Δ^N. However, their definition differs fundamentally from the one presented in this dissertation (they agree only for integer order differences). In 1988, Gray and Zhang [6] introduced the type of fractional difference operator used here; they developed Leibniz formula, a limited composition rule and a version of a power rule for differentiation. However, they dealt exclusively with the nabla (backward) difference

operator and therefore offer results distinct from those presented in this dissertation, where the delta (forward) difference operator is used exclusively.

The information on Partial Differential Equation is valuable for comprehension the subjects viz. Fluid mechanics and heat wave as well as study of soil moisture diffusion, stochastic procedure, population growth, traffic stream, aerodynamics. The investigation of Partial differential conditions has become essential with amazing advances innovation made in science and engineering. Today with the coming of rapid super pcs, calculation has become a focal apparatus of logical advancement. For taking various such condition, we utilize a few numerical strategies and programs. We realize that framework delicate by partial differential equations are compound along these lines it is hard to get simply systematic solution. There are three kinds of fractional differential equations such as Elliptic, Parabolic, Hyperbolic

Consider the general second order partial differential equation proposed by Forsythe and Wasow (1967) is as follws ,

$$A\frac{\partial^2 m}{\partial s^2} + B\frac{\partial^2 m}{\partial s\,\partial y} + C\frac{\partial^2 m}{\partial y^2} + D\frac{\partial m}{\partial s} + E\frac{\partial m}{\partial y} + FW + G(x, y) = 0$$

In two independent variables x, y here the coefficients A, B, C, D, E,F are functions of x and y. the numerical character of the PDE relies on the coefficient A, B and C. the PDE at the point (x_0, y_0) is called Elliptic if $B^2 - 4AC < 0$, Parabolic if $B^2 - 4AC = 0$, Hyperbolic if $B^2 - 4AC > 0$.

In the problems of heat transfer, fluid stream issues and in soil moisture diffusion equation the variable W means a particular variable or specific dependent variable such as soil moisture, temperature, velocity in fluid problems, heat flow, current etc. in flow problems while studding various problems. Fractional calculus is rising field of research. Many issues of various fields are displayed with the assistance of nonlinear partial differential equation. Fractional calculus is a name for the theory of integrals and derivative of arbitrary order unified and generalized the integer- order differential and n-fold integration. Fractional calculus was started when a inquiry arise about the derivative to an integer order n and could be extended to still be valid when n is not an integer.

Preliminaries

The area of fractional calculus is one of numerous applied mathematics disciplines. Instead of integer order, it deals with arbitrarily ordered derivatives and integrals, as well as their applications [7].

Riemann Liouville Fractional Derivative Definition:

For $\alpha \in [n-1, n)$, then Riemann Liouville Fractional Derivative of f of order α is defined as [7-10]

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n} - \alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}\,dx$$

Caputo's Fractional Derivative Definition:

For $\alpha \in [n-1, n)$, then Caputo Fractional Derivative of f of order α is defined as [7-10]

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx$$

Problem 1.

Consider the following initial-value problem, $D_0^{\frac{4}{3}} y(t) - \lambda\, y(t) = t^2$, subjected to the initial conditions,

$$D_0^{\frac{1}{3}} y(t)|_{t=0} = 1 \text{ and } D_0^{\frac{-1}{6}} y(t)|_{t=0} = 2.$$

After simplifying the above fractional differential IVP, the solution of given IVP is,

$$y(t) = t^{\frac{1}{3}} E_{\frac{4}{3},\frac{4}{3}}(\lambda t^{\frac{4}{3}}) + 2\, t^{\frac{-1}{6}} E_{\frac{4}{3},\frac{5}{6}}(\lambda t^{\frac{4}{3}}) + 3t^{\frac{-5}{6}} E_{\frac{4}{3},\frac{1}{6}}(\lambda t^{\frac{4}{3}}) + 2t^{\frac{10}{3}} E_{\frac{4}{3},\frac{13}{3}}(\lambda t^{\frac{4}{3}})$$

Caputo α-th Fractional Sum:

We know the notion of fractional difference in the sense of Caputo for the different notions of fractional sums

Let $\alpha > 0$ and $a \in \mathbb{R}$. The α-th fractional sum of a function f is defined by

$$\Delta_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s),$$

Where $t \in \mathbb{N}_{a+\alpha}$ and $t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t-\alpha+1)}$.

Atici and Eloe Fractional Difference Definition:

The following definition was proposed by Atici and Eloe in 2009.

Let $\alpha > 0$. For any given positive real number a, the α-th fractional sum of a function f is

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t-s-1)^{\overline{\alpha-1}} f(s),$$

Where $t \in \mathbb{N}_{a+\alpha}$ and $t^{\overline{(\alpha)}} = \frac{\Gamma(t+1)}{\Gamma(t-\alpha+1)}$.

The following definition of the (backward nabla) discrete fractional sum was proposed by Grey and Zhang in 1998.

The (nabla) left fractional sum of $\alpha > 0$ (starting from a) is defined by

$${}_{GZ}\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-s-1)^{\overline{\alpha-1}} f(s),$$

Where $t \in \mathbb{N}_{a+1}$.

Let $\alpha > 0$ and α-th fractional Riemann-Liouville like difference is defined by

$\Delta_a^{-\alpha} f(t) = \Delta_{a+(m-\alpha)}^{m} \left(\Delta_a^{-(m-\alpha)} f\right)(t), \quad t \in \mathbb{N}_{a+(m-\alpha)}$,

Where $m-1<\alpha<m, m=\lceil\alpha\rceil$. Here $\lceil . \rceil$ denotes the ceiling number i.e the smallest interger not less than α.

Let $f: \mathbb{N}_0 \rightarrow X$ be a vector-valued sequence and $\alpha>0$. The α-th fractional sum of f, denoted by $\Delta^{-\alpha} f(n) \coloneqq \sum_{j=0}^{n} k^{\alpha}(n-j) f(j),\ n \in \mathbb{N}_0$,
Where

$$k^{\alpha}(j)=\frac{\Gamma(\alpha+j)}{\Gamma(\alpha)\Gamma(j+1)}, \qquad j \in \mathbb{N}_0.$$

The function $\Delta^{-\alpha}: s(\mathbb{N}_0; X) \rightarrow s(\mathbb{N}_0; X)$ is called the fractional delta operator.
Now, we present the definition of fractional differences. The next concept, given by Miller and Ross in 1989, is analogous to the definition of a fractional derivative in the sense of Riemann-Liouville.

Riemann-Liouville fractional Difference Definition:

Let $\alpha>0$. The α-th fractional Riemann-Liouville like difference is defined by

$$\Delta_a^{\alpha} f(t)=\Delta_{a+(m-\alpha)}^{m}\left(\Delta_a^{-(m-\alpha)} f\right)(t),\ t \in \mathbb{N}_{a+(m-\alpha)},$$

Where $m-1<\alpha<m,\ m=\lceil\alpha\rceil$.

Analogously, following the sense of Riemann-Liouville described above, the next definition appears given by Abdeljawad and Atici in 2012
The nabla (left) fractional difference operator of order α > 0 is defined by

$${}_{GZ}\nabla_a^{-\alpha} f(t)=\Delta_{a+1}^{m}\left({}_{GZ}\nabla_a^{(m-\alpha)}\right)(t),\ t \in \mathbb{N}_{a+1}$$

Where $m-1<\alpha<m,\ m=\lceil\alpha\rceil$.

Problem 2.
The $\frac{1}{2}$-fractional difference of a constant sequence $1 \equiv k^1(j)$ is not zero. Indeed,

$$\Delta^{\frac{1}{2}} k^1(n)=k^{\frac{1}{2}}(n+1)=\frac{(2n+2)!}{4^{n+1}\left((n+1)!\right)^2}, \qquad n \in N_0$$

Some Important Properties of Fractional Difference:
Let α > 0. The α-th fractional Caputo like difference is defined by
$(a)\ C\Delta_a^{\alpha} f(t)=\Delta_a^{-(m-\alpha)}(\Delta_a^{m} f)(t), \qquad t \in N_{a+(m-\alpha)};$
$(b)\ CGZ\Delta_a^{\alpha} f(t)=GZ\Delta_a^{-(m-\alpha)}(\Delta_{a+1}^{m} f)(t),\ t \in N_{a+1};$
$(c)\ C\Delta_a^{\alpha} f(t)=\Delta_a^{-(m-\alpha)}(\Delta_a^{m} f)(t), \quad t \in N_a;$
Where $m-1<\alpha<m,\ m=\lceil\alpha\rceil$.
We define the fractional difference (in the sense of Caputo) of order α > 0 by $C\Delta^{\alpha} f(n) \coloneqq \Delta^{-(m-\alpha)}(\Delta^{m} f)(n),\ n \in N_0$, where $m-1<\alpha<m, m=\lceil\alpha\rceil$.

II. Review of literature and development in the subject area

Kuttner was the first derived one studied the time differences of fractional order [12]. Diaz and Osler introduced a discrete fractional difference operator defined as an infinite series [5]. Grey and Zhang developed a fractional calculus for the discrete nabla difference operator [6]. At the same time, Miller and Ross defined a fractional sum via the solution of a linear difference equation [13]. Atici and Eloe introduced the Riemann–Liouville like fractional difference, and developed some of its properties that allow one to obtain solutions of certain fractional difference equations [14]. Ferreira introduced the concept of left and right fractional sum/difference and started a fractional discrete-time theory of the calculus of variations [15,16]. Holm developed and applied the tools of discrete fractional calculus to the area of fractional difference equations [17,18]. Abdeljawad obtained dual identities in fractional difference calculus which they relate the delta and nabla and the left and right fractional sums and differences [19–21]. Goodrich and Peterson develop basic theoretical results in the field of discrete fractional calculus [23]. Moreover, Goodrich studied existence of positive solutions for discrete fractional systems and geometrical properties [24, 25]. Recently, some authors derived some important results on fractional difference and some which can be found in [26-29].

III. Objectives of Proposed Research Work:

1) To study ordinary and partial differential and difference equations.
2) Analysis of different methods for difference equations which are available in the literature.
3) To study basic concept of fractional discrete calculus.
4) To study fractional difference equation.
5) Analysis of development in results for fractional difference equations.
6) To develop different transforms which are used to fractional linear and nonlinear difference equations.
7) To analysis of solution fractional difference IVPs, BVPs and system of difference equations
8) More than two scientific papers are expected to be published in international journals. To study recent applications of fractional difference equations.
9) The results will be presented at least at two international conferences with few dedicated workshops, and networking environment and opportunities to share the scientific results with the European community.
10) Participation to scientific workshops, meetings, specialized training and short-term research visits are expected during the Ph.D. project.
11) The proposed work is to prove that how mathematics is useful in other branches of science. We hope that our research work will be useful in applied sciences and engineering field.
12) Hence, in the present work, we will try to develop the new results and new iterative techniques for fractional difference equation and to obtain their solution. These will have represented graphically by Mathematica, Matlab, Python, Scilab, Mathcad.

IV. Research Methodology:

The following steps will be including in the methodology.

1) To refer the relevant reference books on differential and difference equation and fractional discrete calculus.
2) To study Mathematica, Matlab, Python, Scilab, Mathcad software's.
3) To study fractional difference equations.
4) To study Iterative / Numerical methods for obtaining approximate solutions of fractional difference equations.
5) To discuss the stability and convergence of numerical methods.
6) To represent numerical solutions of fractional difference equations graphically by Mathematica, Matlab, Python, Scilab, Mathcad and discuss their nature.

V. Importance of the Study:

Fractional calculus is becoming a useful and key tool in the analysis of scientific problems in Physics, Engineering, Biology and Economics, Fractional formalism to be useful in solving practical problems, fractional calculus, summation and difference of an arbitrary or fractional order, provides new tools that expand the descriptive power of calculus beyond the familiar integer-order concept of rate of change and area under the curve.

a) International Status: Many scientists and researchers many more contributed in the development of fractional differential equations and Fractional calculus like, Leibniz, Euler, Lagrance, Laplace, Fourier, Abel, Liouvielli, Grunwald, Riemann, Riesz, Miller, Ross, Igor Prodlubny, Richtmeyer, Morton, Smith, Anderson, Roache. Lynn R. Willams (1979) proved the existence of multiple fixed point theorems of positive, completely continuous operators on functional spaces which are used in chemical reactor theory. Nguyen Huu Denn (1993) proved a fixed common point theorem for the pair of mappings satisfying both the Banach contraction principle and Caristi's condition in a complete metric space. Holm developed and applied the tools of discrete fractional calculus to the area of fractional difference equations. He also investigated the class of compactible mappings and establish some fixed point theorems for these mappings on the Meir-Keeler condition. Blazej Szmanda (1996), obtained criterion for the oscillatory and asymptotic behavior of solutions of nonlinear difference equation.

b) National Status: Indian's also contributed remarkable in the development of fractional calculus like R.K. Sexana, V. Srivastava, K.N. Rai, S. Das, K. Gupta, S. Saha Roy, V.B.L. Chawrasia, D. Kumar, Pradeep Kumar, D. Bahuguna, Aalam, Suneel Kumar, Sunny Chauhan and Pant. S.P. Acharya worked on some results on fixed point in uniform space. S.L. Sing from Haridwar, R.P. Pant from Nainatal, D.N Chaliszar from Gujrat, and many more from India had worked on fixed point theorems and its various applications.

Regional Status: Many researchers from Maharashtra have been actively involved in the study of fractional calculus. Like Kiran, M. Kolwankar, A.K. Pani, PSV Nataraj, S. Bhalekar, V. Daftardar- Gejji, K.C. Takale, J. Verma, D.B. Dhaigude, K.L. Bondar, V.C. Borkar and

S.T. Patil (2010), obtained common fixed point theorems in Dynamic programming. They also obtained fixed point theorems with nonlinear alternative via homotopic mapping, comparison theorems for nonlinear difference equations, existence and comparison results for difference ϕ-Laplacian boundary value problems. P.U. Chopade, K.L. Bondar, V.C. Borkar and S.T. Patil (2010-2011) obtained common fixed points for nonexpansive type mapping with application to best approximation, existence of solutions to boundary value problems of functional difference equations. K.L. Bondar, A.B. Jadhav and S.T. Patil (2011-2013) obtained existence of coupled solutions for mixed monotone operator, coupled coincidence point theorem for nonlinear contraction in partially ordered metric space, existence of solutions for nonlinear difference equations with nonlocal conditions, Behavior of solutions of difference equations, local and global existence of solutions for summation equations using fixed point theorem and some common fixed point theorems for the pairs of weakly compactible mappings.

c) Scope of Study:

1) In recent years, there has been a great deal of interest in fractional differential and difference equations. During the last decade, fractional calculus has been applied to almost every field of Science, Engineering and Mathematics. Many application of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems and astrophysics. There are few iterative methods for obtaining the solution of fractional difference equation (FDE), therefore, we develop new existence and uniqueness results to obtain their solutions and discuss the stability and more results.

2) Although a number of researchers have developed new operators involving several special functions, study of their properties, introduce alternate definitions and generalizations of these operators but still there are several functions like I-function, Aleph function, Mittag-Leffler function etc., involving those, one can define a new one.

3) Several researchers have given Integral representations involving operators, special functions and their products. As a large number of operators and special functions are defined by the researchers so various integral presentations and integral formulas can be derived by using these new operators and special functions.

4) Even many Transformations and their applications has been developed by researchers for solving many problems of science and technology, but an easy, suitable and number of alternate approaches is desired to solve different problems of different domains, so there is a good scope of development in this area.

5) The fractional calculus finds use in many fields of science and engineering including the fluid flow, rheology, quantitative biology, electro-chemistry, scattering theory, diffusion transport theory, chemical physics and statistical probability theory, potential theory and many branches of mathematical analysis, like sum and difference equations, operational calculus and univalent function theory, dynamical systems theory, tuning of PID controllers, heat diffusion, circuit synthesis etc. Virtually no area of classical and modern analysis is left untouched by fractional calculus.

6) A significantly large number of earlier works on the subject of fractional calculus give

an interesting account of the theory and applications of fractional calculus operators in many different areas of mathematical analysis like ordinary and partial differential equations, integral equations, special functions, summation of series, etc.

7) The fractional integral operators involving various special functions, have found significant importance and applications in various sub-field of applicable mathematical analysis.

8) The Some transformations are highly useful in study of a wide variety of problems in Mathematics, Physics, Astronomy as well as in Engineering and Technology.

Expected Outcome:

1) We will try to analyze the development in new results and iterative methods for fractional difference equations.

2) We will try to analyze the numerical methods for fractional difference equation.

3) To find the approximate solutions of fractional difference equation by old and new technique and it will have represented graphical by different softwares like Mathematica, Matlab, Python, Scilab, Mathcad.

VI. Purpose of carrying out proposed research work:

We will try to present a full and rigorous theory for composing fractional sum and difference operators, including correcting and broadening the power rule stated incorrectly in some references. We then apply these rules to solve a certain fractional difference initial value problem. We develop some Fractional Transform Method (paying close attention to correct domains of convergence) and apply this to resolve the fractional initial value problem. Then we shift our attention to a certain fractional boundary value problem of arbitrary real order. Specifically, we take N-1 boundary conditions at the left endpoint and 1 boundary condition at the right endpoint and show the existence of a solution using two well-known fixed point theorems.

Difference equations play a major role in Mathematics, Physics and Engineering. The purpose of the research will to discuss the solution of different models by using different transforms and numerical methods. Also, we will represent graphically. The solution of fractional differential equations obtains by different methods using mathematical software's Mathematica, Matlab, Python, Scilab, Mathcad. Also we will study Numerical methods to obtain the solutions of fractional partial Difference equations, such as Explicit methods and finite difference method. We will compare the solutions obtain by different methods.

- **Year wise plan of work and targets to be achieved:**

Sr.No.	Period	Plan of work
1	First year	Study of background information relevant to the research issue. To survey the number of reputed research papers in fractional calculus. To visit reputed libraries
2	Second year	Formation of problems. To attend national, International Conferences. and presentation of research papers
3	Third year	To develop and new approach for obtain the solution of fractional difference equations. publication of research papers.
4	Fourth year	Publishing some papers in reputed journal. To submit synopsis and Thesis

VII. References:

1) S. Elaydi, An Introduction to Difference Equations, Second Edition, Springer New York, NY, 1999.
2) R. R. Agrawal, Difference equations and inequalities, Marcel Dekker, New York, 1992.
3) V. Lakshmikantham, D. Trigiante, Theory of Difference Equations Numerical Mehtods and Applications, First Edition, 1988.
4) Kelley W. and Peterson A., Difference Equations: An Introduction with Applications, Second Edition, Academic Press, New York, New York, 2000.
5) J. B. Diaz and TJ Olser, Differences of Fractional Order, Mathematics of Computation, 28, 1974.
6) H Gray and N Zhang, On a New Definition of the Fractional Difference, Mathematics of Computation, 50, 513-529, 1988.
7) I. Podubny Fractional Differential Equations. Academics press, New York,1999.
8) A.A. Kilbas, H.M. Srivastva and J.J. Trajillo, Theory and applications of fractional differential equations. First edition, 2006.
9) K.S. Miller, B. Ross, An introduction to fractional calculus and fractional differential equation, Newyork,1993.
10) Hilfer R, Application of fractional calculus in Physics, world scientific, Singapore (2000).
11) A. R. Mitchell, D. F. Griffiths, The Finite Difference Method in Partial Differential Equations, John Wiley & Sons, 1980.

12) B Kuttner, On Differences of Fractional Order, Proc. LMS, 7, 453-466, 1957.

13) KS Miller and B Ross, Fractional Difference Calculus, Proceedings of the International Symposium on Univalent Functions, Fractional Calculus and their Applications, Nihon University, Koriyama, Japan, 139-152, 1988.

14) Atici and Eloe introduced the Riemann-Liouville like fractional difference by using the definition of fractional sum of Miller and Ross, 2007.

15) R. Ferreira, Calculus of Variations on Time Scales and Discrete Fractional Calculus. Ph.D. Thesis Universidade de Aveiro, 2010.

16) R. Ferreira, Existence and uniqueness of solution to discrete fractional boundary value problems of order less than one. J. Differ. Equ. Appl. 19, 712-718, 2013.

17) M. T. Holm. The theory of discrete fractional calculus: development and applications, Ph.D. thesis, University of Nebraska, 2011.

18) M. T. Holm. The Laplace transform in discrete fractional calculus. Computers & Mathematics with Applications, 62 (3), 1591–1601, 2011.

19) T. Abdeljawad. On Riemann and Caputo fractional differences. Comput. Math. Appl. 62, 1602-1611, 2011.

20) T. Abdeljawad and F.M. Atici. On the definitions of nabla fractional operators. Abstract and Applied Analysis, volume 2012, 1-13, 2012.

21) T. Abdeljawad and D. Baleanu, "Fractional differences and integration by parts," Journal of Computational Analysis and Applications, vol. 13, no. 3, pp. 574–582, 2011.

22) C. S. Goodrich. Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions. Comput. Math. Appl. 61, 191–202, 2011.

23) C. S. Goodrich. Existence of a positive solution to a system of discrete fractional boundary value problems. Appl. Math. Comput. 217, 4740-4753, 2011.

24) Jia Baoguoa et al. The Relation Between Nabla Fractional Differences and Nabla Integer Differences, by Faculty of Sciences and Mathematics, University of Nis, Serbia, 31(6) 1741–1753, 2017.

25) C. S. Goodrich. On a first-order semipositone discrete fractional boundary value problem. Arch. Math. 99, 509-518, 2018.

26) Syed Sabyel Haider, et al., On Hilfer fractional difference operator, Advances in Difference Equations, 122(2), pp 1-20, 2019.

27) Okkes Ozturk, A Study on nabla discrete fractional operator in mass spring-damper system, New Trends in Mathematical Sciences Vol. 4, No. 4, pp. 137-144, 2020.

28) Athasit Wongcharoen, et al., Nonlinear Fractional q-Difference Equation with Fractional Hadamard and Quantum Integral Nonlocal Conditions, Hindawi Journal of Function Spaces, Volume 2020, 10 (2020).

29) Alberto C., Nikolay, J. J. Mohan, Non-Trivial Solutions of Non-Autonomous Nabla Fractional Difference Boundary Value Problems, symmetry, Vol. 13, pp.1-15, 2021.

Mr. Yogesh H. Shirole
(Research Scholar)

Dr. Suryakant M. Jogdand
(Research Guide)